



Falcón-Cortés, A., Boyer, D., Giuggioli, L., & Majumdar, S. N. (2017). Localization Transition Induced by Learning in Random Searches. *Physical Review Letters*, 119(14), [40603].  
<https://doi.org/10.1103/PhysRevLett.119.140603>

Peer reviewed version

Link to published version (if available):  
[10.1103/PhysRevLett.119.140603](https://doi.org/10.1103/PhysRevLett.119.140603)

[Link to publication record in Explore Bristol Research](#)  
PDF-document

This is the author accepted manuscript (AAM). The final published version (version of record) is available online via APS at <https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.119.140603> . Please refer to any applicable terms of use of the publisher.

## University of Bristol - Explore Bristol Research

### General rights

This document is made available in accordance with publisher policies. Please cite only the published version using the reference above. Full terms of use are available:  
<http://www.bristol.ac.uk/red/research-policy/pure/user-guides/ebr-terms/>

# Supplemental Material

## I. NO-RETURN PROBABILITY FOR LÉVY FLIGHTS: RECURRENT VS. TRANSIENT BEHAVIOR

Consider a  $d$ -dimensional Euclidean lattice. A random walker moves on the sites of this lattice with random jumps at each time step. The jump lengths are independent and identically distributed (i.i.d) random variables drawn from a normalized distribution  $p(\ell)$ . The walker starts at some arbitrary site ( $\mathbf{x}_0$  at time  $t = 0$ ). Then the probability of no return to the initial site is given by the well known formula

$$P_{no-return} = \frac{1}{\int_{\mathcal{B}} \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{1}{1-\tilde{p}(\mathbf{k})}} \quad (1)$$

where  $\tilde{p}(\mathbf{k})$  is the Fourier transform of the jump distribution

$$\tilde{p}(\mathbf{k}) = \sum_{\ell} p(\ell) e^{-i\mathbf{k} \cdot \ell}. \quad (2)$$

Thus  $P_{no-return}$  in Eq. (1) is nonzero or zero depending on whether the integral in the denominator is finite or divergent. The divergence of this integral depends on the small  $|\mathbf{k}|$  behavior of  $\tilde{p}(\mathbf{k})$ . In general, for Lévy flights, the small  $k$  behavior is given by

$$\tilde{p}(\mathbf{k}) \simeq 1 - K_{\mu} |\mathbf{k}|^{\mu} \quad (3)$$

where the Lévy index  $0 < \mu \leq 2$ . For  $\mu < 2$ , the second moment of the jump distribution is divergent, while for  $\mu = 2$ , the second moment is finite. Hence, standard Euclidean random walks with nearest neighbour jumps correspond to  $\mu = 2$ , with  $\tilde{p}(\mathbf{k}) = 1 - D_0 |\mathbf{k}|^2$ . From now on, we will consider the general  $0 < \mu \leq 2$  case, and it will include the  $\mu = 2$  case corresponding to standard nearest neighbour random walks. Substituting the small  $k$  behavior in the integral in the denominator of Eq. (1), it is evident that this integral diverges if  $d < \mu$  and is finite if  $d > \mu$ . Thus, for  $d < \mu$ ,  $P_{no-return} = 0$ , while it is non zero for  $d > \mu$ . Thus, for Lévy flights with index  $\mu$  ( $0 < \mu \leq 2$ ), the critical dimension is  $d_c = \mu$  that separates the recurrent ( $d < \mu$ ) behavior from the transient ( $d > \mu$ ) behavior. For ordinary random walks ( $\mu = 2$ ),  $d_c = 2$ .

## II. CRITICAL BEHAVIOR OF THE ORDER PARAMETER $P_0$

We first consider the critical value  $q_c$  (for fixed  $\gamma$ ) that separates the delocalised phase with  $P_0 = 0$  for  $q < q_c$  and the localised phase with  $P_0 > 0$  for  $q > q_c$ . In the main text, we have shown that the value of  $q_c$  is given by the formula

$$q_c = \frac{(1-\gamma)P_{no-return}}{\gamma + (1-\gamma)P_{no-return}} \quad (4)$$

where  $P_{no-return}$  is given in Eq. (1). So, clearly, for Lévy flights with index  $0 < \mu \leq 2$  (including standard random walks corresponding to  $\mu = 2$ ), using results on  $P_{no-return}$  from the previous Section I, we have

$$q_c = \frac{(1-\gamma)P_{no-return}}{\gamma + (1-\gamma)P_{no-return}} > 0 \quad \text{for } d > \mu \quad (5)$$

$$= 0 \quad \text{for } d < \mu. \quad (6)$$

We now consider how  $P_0$  increases from its value 0 as  $q$  increases above  $q_c$ . We want to show here that in general, as  $q \rightarrow q_c^+$ ,

$$P_0 \sim (q - q_c)^{\beta} \quad (7)$$

where the exponent  $\beta$  depends continuously on  $\mu$  and  $d$  in the  $\mu - d$  plane. We will show below that

$$\beta = \begin{cases} 1 & \text{for } d > 2\mu \\ \frac{\mu}{d-\mu} & \text{for } \mu < d < 2\mu \\ \frac{d}{\mu-d} & \text{for } d < \mu \end{cases} \quad (8)$$

where, we recall, that in the last case ( $d < \mu$ ),  $q_c = 0$ .

To derive this result for  $\beta$ , we start from the equation in the main text that determines  $P_0$  for any given  $q$ , namely

$$\frac{1}{(2\pi)^d} \int_{\mathcal{B}} \frac{d^d \mathbf{k}}{(1-q)[1-\tilde{p}(\mathbf{k})] + q\gamma P_0} = \frac{1-\gamma}{q\gamma(1-\gamma P_0)}. \quad (9)$$

Of course, at  $q = q_c$ ,  $P_0 = 0$  and this gives us

$$\frac{1}{(2\pi)^d} \int_{\mathcal{B}} \frac{d^d \mathbf{k}}{(1-q_c)[1-\tilde{p}(\mathbf{k})]} = \frac{1-\gamma}{q_c \gamma}, \quad (10)$$

which indeed leads to the expression for  $q_c$  in Eq. (4).

We are now ready to see how  $P_0$  increases from 0 as  $q$  increases above  $q_c$ . For this we consider two cases separately.

**Case I:  $q_c > 0$ .** As we have seen before, this corresponds to the transient regime where  $P_{no-return} > 0$ . For Lévy flights, this means  $d > d_c = \mu$ . To proceed, we first subtract Eq. (9) from Eq. (10) which gives

$$\int_{\mathcal{B}} \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{[q\gamma\delta - (q - q_c)(1 - \tilde{p}(\mathbf{k}))]}{(1 - \tilde{p}(\mathbf{k}))[(1 - q)(1 - \tilde{p}(\mathbf{k})) + q\gamma P_0]} = \frac{(1 - q_c)(1 - \gamma)(q - q_c - q\gamma P_0)}{qq_c\gamma(1 - \gamma P_0)}. \quad (11)$$

We then set  $q = q_c + \epsilon$  with  $\epsilon \rightarrow 0$  and  $P_0 = \delta$  with  $\delta \rightarrow 0$ . Our goal is to find how  $\delta$  scales with  $\epsilon$  to leading order in small  $\epsilon$ . In this limit, the leading contribution to the integral on the left hand side (lhs) of Eq. (11) comes from the small  $k$  region, where we can replace  $\tilde{p}(\mathbf{k})$  by Eq. (3). Keeping only the leading order terms and simplifying, we obtain

$$\delta I(\delta) + O(\delta) = A\epsilon \quad (12)$$

where  $A = (1 - \gamma)(1 - q_c)K_\mu^2/(\gamma^2 q_c^3)$  is just a constant and  $I(\delta)$  is the integral

$$I(\delta) = \int_{\mathcal{B}} \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{1}{|\mathbf{k}|^\mu [|\mathbf{k}|^\mu + b\delta]} \quad (13)$$

where  $b = q_c\gamma/(K_\mu(1 - q_c))$  is a constant. We now need to analyse the integral  $I(\delta)$  as  $\delta \rightarrow 0$ . There are again two cases: (1)  $d > 2\mu$  and (2)  $\mu < d < 2\mu$ . We consider them separately.

1.  $d > 2\mu$ : In this case, if we put  $\delta = 0$  in  $I(\delta)$  in Eq. (13), the integral converges as  $k \rightarrow 0$ , making  $I(0)$  finite. Hence, from Eq. (12), we get

$$\delta \sim \epsilon \quad \text{implying} \quad \beta = 1 \quad \text{for} \quad d > 2\mu. \quad (14)$$

2.  $\mu < d < 2\mu$ : In this case, the integral  $I(0)$  in Eq. (13) is divergent. Hence, to extract the leading singularity, we rescale  $k \rightarrow \delta^{1/\mu} y$  in Eq. (13).

$$I(\delta) \sim \delta^{\frac{d}{\mu}-2} \int_0^\infty \frac{dy y^{d-1-\mu}}{y^\mu + b}. \quad (15)$$

Note that the integral in Eq. (15) is convergent in both limits  $y \rightarrow 0$  and  $y \rightarrow \infty$ , as long as  $\mu < d < 2\mu$ . Hence, substituting Eq. (15) in Eq. (12) we get, to leading order

$$\delta \sim \epsilon^{\frac{\mu}{d-\mu}} \quad \text{implying} \quad \beta = \frac{\mu}{d-\mu} \quad \text{for} \quad \mu < d < 2\mu. \quad (16)$$

**Case II:  $q_c = 0$ .** This case corresponds to the recurrent case when  $P_{no-return} = 0$ , making  $q_c = 0$ . As discussed before, for Lévy flights with index  $0 < \mu \leq 2$ , this happens when  $d < d_c = \mu$ . In this case we analyse directly Eq. (9) by substituting  $q = \epsilon$  and  $P_0 = \delta$ . Again, keeping only the small  $\mathbf{k}$  contribution to the integral, we get to leading order

$$\int_{\mathcal{B}} \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{1}{|\mathbf{k}|^\mu + \epsilon\delta} \sim \frac{1}{\epsilon} \quad (17)$$

Rescaling  $k = (\epsilon\delta)^{1/\mu}y$  gives

$$(\epsilon\delta)^{\frac{d}{\mu}-1} \int_0^\infty \frac{dy y^{d-1}}{y^\mu + 1} \sim \frac{1}{\epsilon}. \quad (18)$$

Note that the integral in Eq. (18) is convergent in both limit  $y \rightarrow 0$  and  $y \rightarrow \infty$  for  $0 < d < \mu$ . Hence, Eq. (18) then gives

$$\delta \sim \epsilon^{\frac{d}{\mu-d}} \quad \text{implying} \quad \beta = \frac{\mu}{\mu-d} \quad \text{for} \quad 0 < d < \mu. \quad (19)$$

This completes the derivation of the result for the exponent  $\beta$  given in Eqs. (8), (8) and (8).

### III. LOCALIZATION OF THE $1d$ RANDOM WALK WITH NEAREST NEIGHBORS JUMPS

We derive here an analytical expression for the stationary distribution  $P_n$ . We consider the particular case of the random walk with nearest neighbor jumps in one dimension, where the step distribution is given by  $p(l) = \frac{1}{2}[\delta_{l,1} + \delta_{l,-1}]$ . The Fourier transform of  $p(l)$  is  $\tilde{p}(k) = \cos k$ . In this case, the expression given by Eq. (4) of the main text for the Fourier transform of  $P_n$  becomes

$$\tilde{P}(k) = \frac{\gamma P_0 [1 - (1-q)\cos k]}{(1-q)(1-\cos k) + q\gamma P_0} = \gamma P_0 + \frac{q\gamma P_0 (1 - \gamma P_0)}{(1-q)(1-\cos k) + q\gamma P_0}. \quad (20)$$

The form of the steady-state probability can be derived by inverse Fourier transforming. Using the fact that for  $a^2 > 1$  [1]:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dk \frac{\cos(kn)}{1 + a^2 - 2a \cos k} = \frac{1}{(a^2 - 1)a^{|n|}}, \quad (21)$$

we write the denominator  $(1-q)(1-\cos k) + q\gamma P_0$  under the form  $b(1 + a^2 - 2a \cos k)$ . By identification, we have:

$$2ab = 1 - q \quad (22)$$

$$b(1 + a^2) = 1 - q(1 - \gamma P_0) \quad (23)$$

which yields

$$a = 1 + \frac{\gamma q P_0}{1 - q} + \sqrt{\frac{\gamma q P_0}{1 - q} \left( 2 + \frac{\gamma q P_0}{1 - q} \right)}. \quad (24)$$

Using Eq. (21) and (22), the inversion of Eq. (20) gives:

$$P_n = \gamma P_0 \delta_{n,0} + \frac{q\gamma P_0 (1 - \gamma P_0)}{1 - q} \frac{2a}{(a^2 - 1)a^{|n|}}. \quad (25)$$

By evaluating Eq. (25) at  $n = 0$ , the above expression can be rewritten in compact form:

$$P_n = \gamma P_0 \delta_{n,0} + (1 - \gamma) P_0 a^{-|n|}, \quad (26)$$

which is one of the main result of this section. We are only left with the determination of  $P_0$ , the asymptotic probability of occupying the inhomogeneity. For this purpose, we evaluate once more Eq. (25) at  $n = 0$ , obtaining:

$$2q\gamma(1 - \gamma P_0) = (1 - \gamma)(1 - q)(a - a^{-1}). \quad (27)$$

Inserting the expression of  $a$  given by Eq. (24) into Eq. (27) gives a quadratic equation for  $P_0$  whose only positive root is

$$P_0 = \frac{-(1-q)(1-\gamma)^2 - q\gamma^2}{q\gamma(1-2\gamma)} + \frac{\sqrt{[(1-q)(1-\gamma)^2 + q\gamma^2]^2 + (q\gamma)^2(1-2\gamma)}}{q\gamma(1-2\gamma)}, \quad (28)$$

for  $\gamma \neq 1/2$ . When  $\gamma = 1/2$ , the solution is simply  $P_0 = q$ .

---

[1] Gradshteyn, I. S. and Ryzhik, I. M., *Table of integrals, series, and products*, Eighth ed., (Elsevier/Academic Press, Amsterdam, 2015).